SECTION 4.3: WHAT DERIVATIVES TELL US

We know if f is differentiable at $x = a$ then f is **locally linear** at $x = a$ and $f'(a)$ is the **slope** of the tangent line at the point $(a, f(a))$. In this section, we explore how local behavior near a point can be extrapolated to global behavior over an interval. First, we review some key definitions from algebra.

DEFINITION: Suppose f is a function defined on an interval I. We say f is:

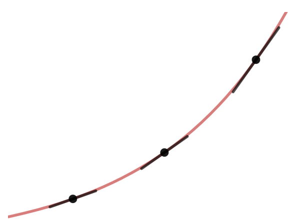
- **increasing** on I if and only if for all real numbers a, b in I with $a < b$, $f(a) < f(b)$. enumerate That is, as your inputs increase, your outputs increase; as you move from left to right, the graph rises.
- **decreasing** on I if and only if for all real numbers a, b in I with $a < b$, $f(a) > f(b)$.

  That is, as your inputs increase, your outputs decrease; as you move from left to right, the graph falls.
- **constant** on I if and only if for all real numbers a, b in I, $f(a) = f(b)$.

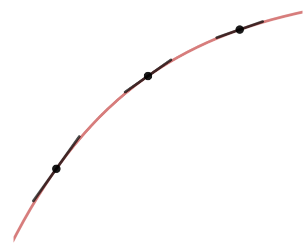
  That is, as your inputs increase, your outputs don't change; the graph is horizontal.

THE FIRST DERIVATIVE AND GRAPHS: Suppose f is differentiable on an open interval I:

- If $f'(x) > 0$ for all x in I, then f is increasing on I. Geometrically. the slopes of all tangents are positive:

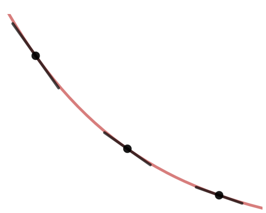


$f'(x) > 0$ for all x in I

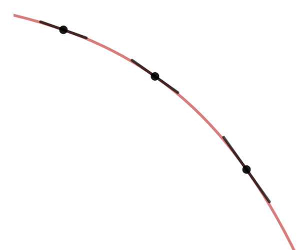


$f'(x) > 0$ for all x in I

- If $f'(x) < 0$ for all x in I, then f is decreasing on I. Geometrically, the slopes of all tangents are negative:



$f'(x) < 0$ for all x in I



$f'(x) < 0$ for all x in I

- If $f'(x) = 0$ for all x in I, then f is constant on I.

$f'(x) = 0$ for all x in I

PROOF: Suppose $f'(x) > 0$ on an interval I. To show f is increasing on I, let a and b be two real numbers in I with $a < b$. We need to show that $f(a) < f(b)$. To accomplish this, we apply the Mean Value Theorem (MVT) to the function f on the interval $[a, b]$.

To say $f'(x) > 0$ for all x in I means, in particular, that $f'(x)$ exists for all x in I. Hence, f is differentiable on the interval $[a, b]$ which means f must be continuous on $[a, b]$ as well. Hence, the MVT applies to f on $[a, b]$.

According to the MVT, there is a number c in the interval (a, b) so that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. Since c is in the interval (a, b), c is also in the interval I so $f'(c) > 0$.

Hence, $\dfrac{f(b) - f(a)}{b - a} = f'(c) > 0$. Since $a < b$, $b - a > 0$ so if $\dfrac{f(b) - f(a)}{b - a} > 0$, then $f(b) - f(a) > 0$, or $f(a) < f(b)$.

We leave it to the interested reader to rework the proof in the other two cases.

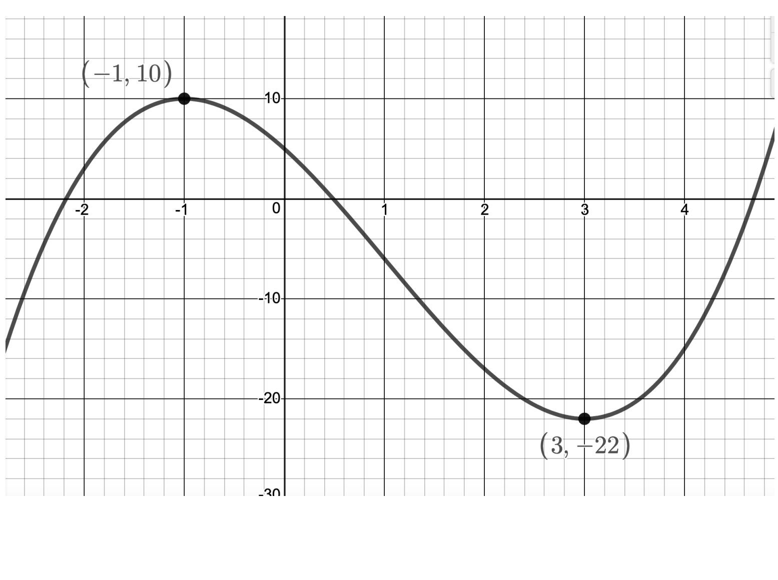
STRATEGY: To find the intervals of increase / decrease / constant for f, make a Sign Diagram for $f'(x)$.

EXAMPLE 1: Find the open intervals over which $f(x) = x^3 - 3x^2 - 9x + 5$ is increasing, decreasing, and constant.

We first find $f'(x) = 3x^2 - 6x - 9$. To make a Sign Diagram for $f'(x)$, we find the critical numbers: places where $f'(x) = 0$ or $f'(x)$ does not exist. Since $f'(x)$ is a polynomial, $f'(x)$ always exists so we focus on solving $f'(x) = 3x^2 - 6x - 9 = 0$. Factoring gives $3(x^2 - 2x - 3) = 0$, and factoring some more gives $3(x - 3)(x + 1) = 0$.
We get two solutions: $x = -1$ and $x = 3$ which divides the x-axis into three regions: $x < -1$, $-1 < x < 3$ and $x > 3$.
To make a Sign Diagram for $f'(x)$, we test the sign of $f'(x)$ in each of the three regions.
We find $f'(-3) = 3(-3)^2 - 6(-3) - 9 = (+)$, $f'(0) = 3(0)^2 - 6(0) - 9 = (-)$ and $f'(4) = 3(4)^2 - 6(4) - 9 = (+)$.
Below on the left is a Sign Diagram for $f'(x)$ and on the right is what this means for the graph of of $y = f(x)$.

$$\begin{array}{cccccccc} & (+) & 0 & (-) & 0 & (+) & f'(x) \\ -\infty & \longleftarrow & + & \text{---} & + & \longrightarrow & \infty \\ & \uparrow & -1 & \uparrow & 3 & \uparrow & x \\ & -3 & & 0 & & 4 & \end{array} \qquad \begin{array}{cccccccc} & \nearrow & \rightarrow & \searrow & \rightarrow & \nearrow & f(x) \\ -\infty & \longleftarrow & + & \text{---} & + & \longrightarrow & \infty \\ & & -1 & & 3 & & x \end{array}$$

We find f is increasing on $(-\infty, -1)$ and again on $(3, \infty)$ while f is decreasing on $(-1, 3)$. At the points $x = -1$ and $x = 3$, we have $f'(x) = 0$ so f is locally flat there.
Since f changes from increasing just to the left of $x = -1$ to decreasing just to the right of $x = -1$, it stands to reason that f has a local maximum at $x = -1$. The local maximum value is $f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 5 = 10$.
Similarly, since f changes from decreasing just to the left of $x = 3$ to increasing just to the right of $x = 3$, f has a local minimum at $x = 3$. The local minimum value is $f(3) = (3)^3 - 3(3)^2 - 9(3) + 5 = -22$.
A quick check using desmos confirms our results.



We generalize our observations about local extrema in the following result.

THE FIRST DERIVATIVE TEST FOR LOCAL EXTREMA:

Let c be a critical number for a continuous function f (i.e., $f'(c) = 0$ or $f'(c)$ does not exist.)

- If $f'(x)$ changes from $(+)$ for $x < c$ to $(-)$ for $x > c$, f has a local maximum at $x = c$.
- If $f'(x)$ changes from $(-)$ for $x < c$ to $(+)$ for $x > c$, f has a local minimum at $x = c$.
- If $f'(x)$ doesn't change sign going from $x < c$ to $x > c$, f does not have a local extremum at $x = c$.

EXAMPLE 2: Let $f(x) = x^{4/3} - 4x^{1/3}$.

1. List the open intervals over which f is increasing, decreasing, and constant.

   We find $f'(x) = \dfrac{4}{3}x^{1/3} - \dfrac{4}{3}x^{-2/3} = \dfrac{4x^{1/3}}{3} - \dfrac{4}{3x^{2/3}} = \dfrac{4x^{1/3}}{3} \cdot \dfrac{x^{2/3}}{x^{2/3}} - \dfrac{4}{3x^{2/3}} = \dfrac{4x}{3x^{2/3}} - \dfrac{4}{3x^{2/3}} = \dfrac{4x-4}{3x^{2/3}}$.

   We see $f'(x) = \dfrac{4x-4}{3x^{2/3}}$ is undefined when $3x^{2/3} = 0$, that is, when $x = 0$.

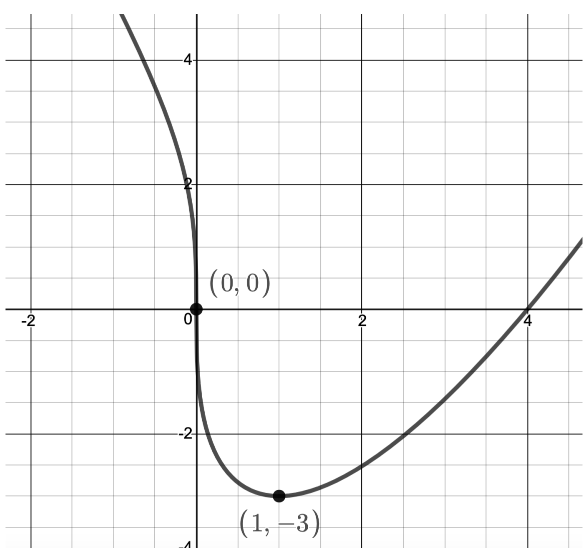
   Solving $f'(x) = \dfrac{4x-4}{3x^{2/3}} = 0$ gives $4x - 4 = 0$ so $x = 1$. Our Sign Diagram for $f'(x)$ is below.

   $$\begin{array}{ccccccccc} & (-) & ? & (-) & 0 & (+) & f'(x) \\ -\infty \longleftarrow & & 0 & & 1 & & \longrightarrow \infty \\ & \uparrow & & \uparrow & & \uparrow & x \\ & -1 & & \tfrac{1}{2} & & 2 & \end{array} \qquad \begin{array}{ccccccc} & \searrow & ? & \searrow & \to & \nearrow & f(x) \\ -\infty \longleftarrow & & 0 & & 1 & & \longrightarrow \infty \\ & & & & & & x \end{array}$$

   We see that f is decreasing for $x < 0$ and again from $0 < x < 1$. Since 0 is in the domain of f, we can connect these two intervals to write that f is decreasing from $(-\infty, 1)$. We see f is increasing from $(1, \infty)$.

2. Locate all the local extrema.

   Since f changes from decreasing just to the left of $x = 1$ to increasing just to the right of $x = 1$, we know f has a local minimum at $x = 1$. The local minimum value is $f(1) = (1)^{4/3} - 4(1)^{1/3} = -3$. Note even though $x = 0$ is a critical value, f doesn't have a local extremum there since f is decreasing through that point. A quick check using desmos reveals that, in fact, we have a vertical tangent at $x = 0$.



EXAMPLE 3: (VIDEO) Let $f(x) = 2\sin(x) - \sin(2x)$ restricted to the interval $[0, 2\pi]$.

1. List the open intervals over which f is increasing, decreasing, and constant.

   First, we find $f'(x)$:

   Ans: $f'(x) = 2\cos(x) - 2\cos(2x)$

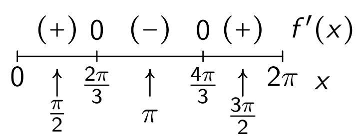
   Next, we solve $f'(x) = 0$. **HELPFUL HINT:** $\cos(2x) = 2\cos^2(x) - 1$

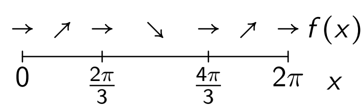
   Ans: $f'(x) = 2\cos(x) - 2\cos(2x) = \ldots = -2(2\cos(x) + 1)(\cos(x) - 1) = 0.$

   In $(0, 2\pi)$, the only solutions are $x = \frac{2\pi}{3}$ and $x = \frac{4\pi}{3}$.

   We also get $\cos(x) - 1 = 0$ or $\cos(x) = 1$ which occurs at the endpoints $x = 0$ and $x = 2\pi$.

   Now, we make a Sign Diagram for $f'(x)$ and interpret it:





   Ans: f is increasing on: $\left(0, \frac{2\pi}{3}\right)$ and $\left(\frac{4\pi}{3}, 2\pi\right)$

   Ans: f is decreasing on: $\left(\frac{2\pi}{3}, \frac{4\pi}{3}\right)$

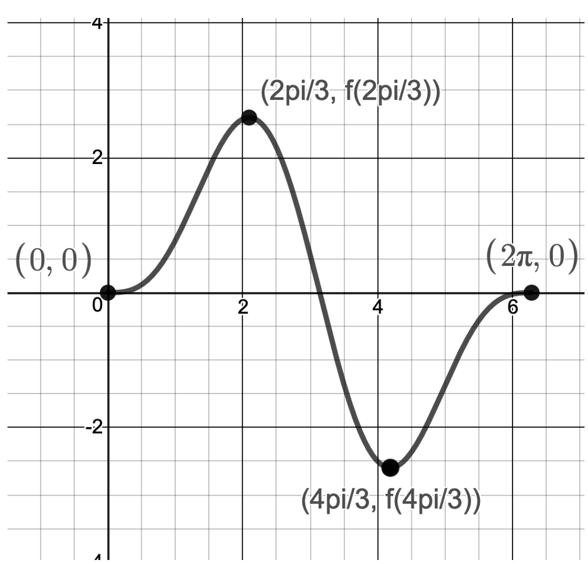
2. Locate all the local extrema.

   Ans:

   f has a local maximum: $\left(\frac{2\pi}{3}, f\left(\frac{2\pi}{3}\right)\right) = \left(\frac{2\pi}{3}, \frac{3\sqrt{3}}{2}\right)$

   f has a local minimum: $\left(\frac{4\pi}{3}, f\left(\frac{4\pi}{3}\right)\right) = \left(\frac{4\pi}{3}, -\frac{3\sqrt{3}}{2}\right)$

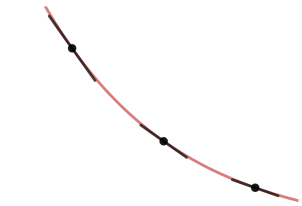
   We check using desmos:



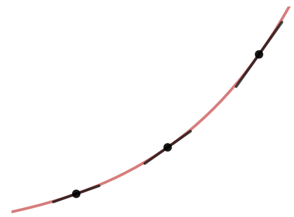
We have seen the second derivative play a role in some important application problems as acceleration, for instance. We may wonder what role $f''(x)$ plays in the graph of a function $y = f(x)$. The key to understanding here is that $f''(x)$ is the derivative of $f'(x)$. Hence, if $f''(x) > 0$, this means the **slopes** $f'(x)$ are **increasing**. Likewise, if $f''(x) < 0$, the **slopes** $f'(x)$ are **decreasing**. This is the notion of **concavity**.

THE SECOND DERIVATIVE AND GRAPHS: Suppose f is twice differentiable on an open interval I:

- If $f''(x) > 0$ for all x in I, then **slopes** are **increasing** and f is **concave up** on I. Geometrically:

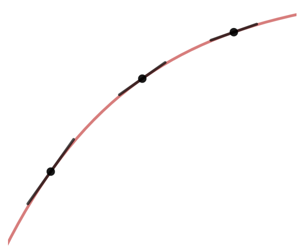


slopes are increasing towards 0

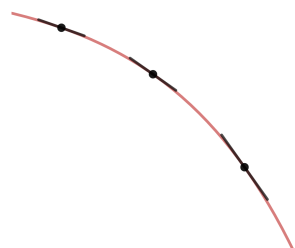


slopes are increasing away from 0

- If $f''(x) < 0$ for all x in I, then **slopes** are **decreasing** and f is **concave down** on I. Geometrically:



slopes are decreasing towards 0

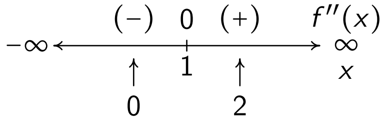


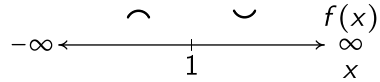
slopes are decreasing away from 0

- If $f''(x) = 0$ for all x in I, then **slopes** are **constant** which the graph of $y = f(x)$ is a line.

EXAMPLE 4: Find the open intervals over which $f(x) = x^3 - 3x^2 - 9x + 5$ is concave up and concave down.

To analyze the concavity of the graph of f, we need to make a Sign Diagram for $f''(x)$.
We have from previous work that $f'(x) = 3x^2 - 6x - 9$. Hence we find $f''(x) = 6x - 6$.
Solving $f''(x) = 6x - 6 = 0$ gives $x = 1$. We find $f''(0) = 6(0) - 6 = (-)$ and $f''(2) = 6(2) - 6 = (+)$.
We have our Sign Diagram below on the left and our interpretation below on the right.

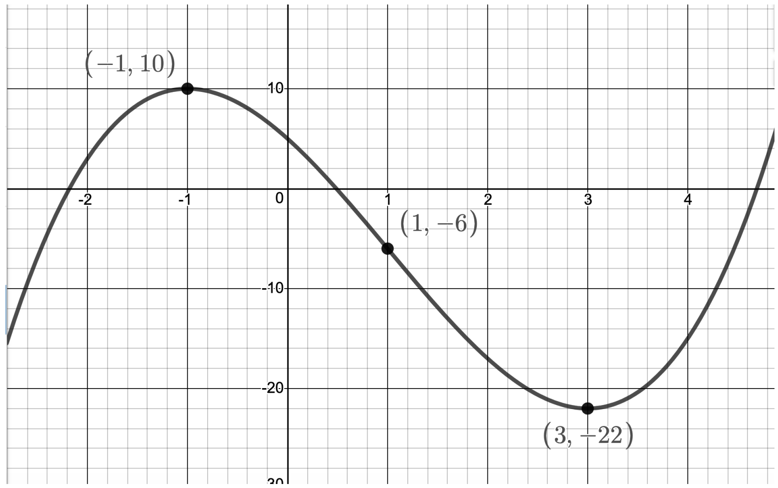




We find f is concave down on $(-\infty, 1)$ and concave up on $(1, \infty)$.

At $x = 1$, the concavity changes. We find $f(1) = (1)^3 - 3(1)^2 - 9(1) + 5 = -6$ and we call the point $(1, -6)$ an **inflection point**. In this case since the concavity changes from concave down to concave up, the point $(1, -6)$ is the point on the graph of $y = f(x)$ were the slopes stop decreasing and start to increase.

A quick check using desmos confirms our results.



Note that we can use concavity to help us distinguish local extrema.

For the function above, both $f'(-1) = 0$ and $f'(3) = 0$. Note that $f''(-1) < 0$ which means f is concave down there. This forces f to have a local maximum at $(-1, 6)$. Likewise, $f''(3) > 0$ which means f is concave up there. This forces f to have a local minimum at $(3, -22)$. We generalize this observation below.

THE SECOND DERIVATIVE TEST FOR LOCAL EXTREMA:

Suppose f is continuous and $f'(c) = 0$.

- If $f''(c) > 0$ then f has a local minimum at $x = c$.
- If $f''(c) < 0$ then f has a local maximum at $x = c$.
- If $f''(c) = 0$ then the test is inconclusive. f may or may not have a local extremum at $x = c$.
  (In this case, we would appeal to the first derivative test.)

EXAMPLE 5: Let $f(x) = x^{4/3} - 4x^{1/3}$.

1. List the open intervals over which f is concave up and concave down.

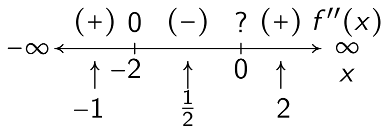
   We know from earlier that $f'(x) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3}$. Hence,

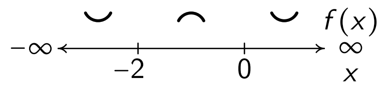
$$f''(x) = \frac{4}{9}x^{-2/3} + \frac{8}{9}x^{-5/3} = \frac{4}{9x^{2/3}} + \frac{8}{9x^{5/3}} = \frac{4}{9x^{2/3}} \cdot \frac{x^{3/3}}{x^{3/3}} + \frac{8}{9x^{5/3}} = \frac{4x}{9x^{5/3}} + \frac{8}{9x^{5/3}} = \frac{4x+8}{9x^{5/3}}$$

   We see $f''(x) = \dfrac{4x+8}{9x^{5/3}}$ is undefined when $9x^{5/3} = 0$, that is, when $x = 0$.

   Solving $f''(x) = \dfrac{4x+8}{9x^{5/3}}$ gives $4x + 8 = 0$ so $x = -2$.

Our Sign Diagram for $f''(x)$ is below.





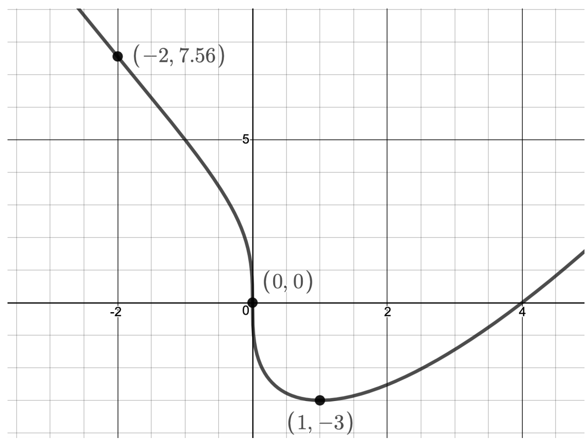
We see f is concave up on $(-\infty,-2)$ and again from $(0,\infty)$. f is concave down on $(-2,0)$.

2. Locate the inflection points.

   Since f changes concavity at both $x=-2$ and $x=0$, we have inflection point at both of these values.

   We find: $f(-2)=(-2)^{4/3}-4(-2)^{1/3}=2(2)^{1/3}+4(2)^{1/3}=6(2)^{1/3}$. So $\left(-2,6(2)^{1/3}\right)$ is one inflection point.

   When $x=0$, $f(0)=(0)^{4/3}-4(0)^{1/3}=0$, so $(0,0)$ is the other inflection point.



Checking with desmos, it's not apparent that the graph of $y=f(x)$ is concave up for $x<-2$. We invite the reader to graph f and zoom out to see that characteristic of the graph.

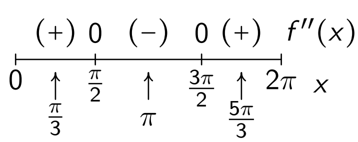
EXAMPLE 6: (VIDEO) Find the inflection points of $f(x)=2\sin(x)-\sin(2x)$ restricted to the interval $[0,2\pi]$.

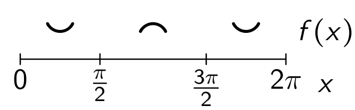
1. First, find $f''(x)$.

   Ans: $f'(x)=1+2\sin(x)$ so $f''(x)=2\cos(x)$.

2. Make a Sign Diagram for $f''(x)$ on $[0,2\pi]$.

   Solving $f''(x)=2\cos(x)=0$ gives $x=\frac{\pi}{2}$ and $x=\frac{3\pi}{2}$.

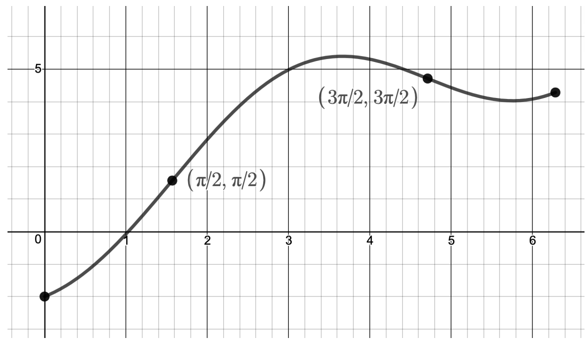




3. Look for points where the concavity changes.

   Inflection points: $\left(\frac{\pi}{2}, f\left(\frac{\pi}{2}\right)\right) = \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2}, f\left(\frac{3\pi}{2}\right)\right) = \left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)$.

   We check using desmos:



EXAMPLE 7: (VIDEO) Let $f(x) = x\sqrt[3]{x-a}$ where $a > 0$.

1. Find $f'(x)$ and use this to find the open intervals over which f is increasing and decreasing.

   Ans: $f'(x) = \dfrac{4x-3a}{3(x-a)^{2/3}}$. Next, we need to make a Sign Diagram for $f'(x)$:

   $f'(x)$ does not exist when $x - a = 0$ or $x = a$; $f'(x) = 0$ when $4x - 3a = 0$ or $x = \dfrac{3a}{4}$:

   $(-)$ 0 $(+)$? $(+)$ $f'(x)$
   $-\infty$ ↑ $\frac{3a}{4}$ ↑ a ↑ ∞ x
   0 0.8a 2a

   ↘ → ↗ ↑ ↗ $f(x)$
   $-\infty$ $\frac{3a}{4}$ a ∞ x

   Ans: We interpret: f is decreasing from $\left(-\infty, \dfrac{3a}{4}\right)$ and increasing from $\left(\dfrac{3a}{4}, \infty\right)$.

2. Locate the local extrema.

   Ans: f has a local (and absolute!) min at $\left(\dfrac{3a}{4}, -\dfrac{3a\sqrt[3]{2a}}{8}\right)$

3. Find $f''(x)$ and use this to find the open intervals over which f is concave up and concave down.

   Ans: $f''(x) = \dfrac{4x-6a}{9(x-a)^{5/3}}$. Next, we make a Sign Diagram for $f''(x)$:

   $f''(x)$ does not exist when $x - a = 0$ or $x = a$; $f''(x) = 0$ when $4x - 6a = 0$ or $x = \dfrac{3a}{2}$:

   $(+)$? $(-)$ 0 $(+)$ $f''(x)$
   $-\infty$ ↑ a ↑ $\frac{3a}{2}$ ↑ ∞ x
   0 1.25a 2a

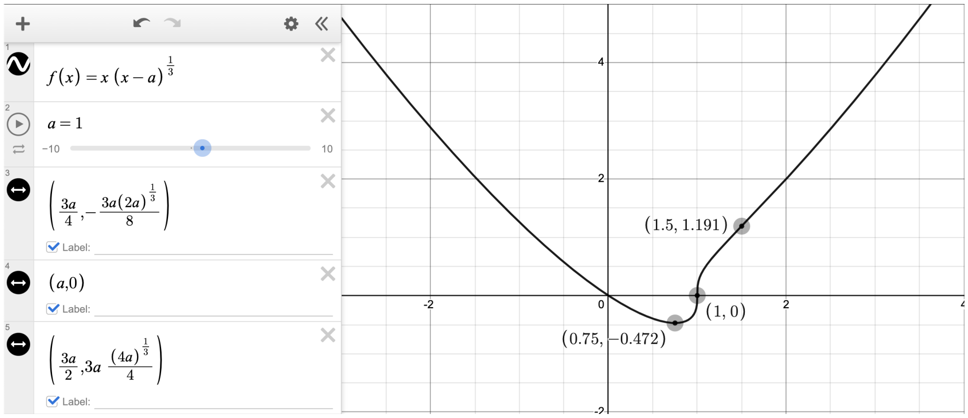
   $\smile$ $\frown$ $\smile$ $f(x)$
   $-\infty$ a $\frac{3a}{2}$ ∞ x

   Ans: We interpret: the graph of f is: concave up on: $(-\infty, a)$, $\left(\dfrac{3a}{2}, \infty\right)$; concave down on: $\left(a, \dfrac{3a}{2}\right)$.

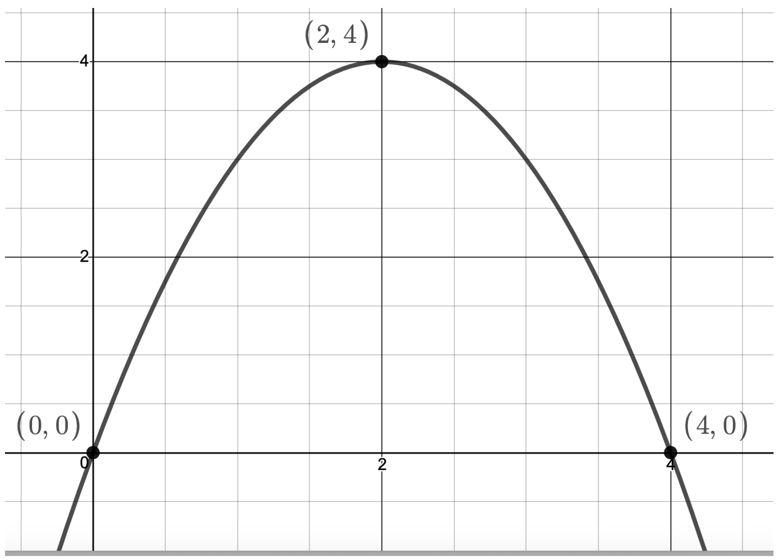
4. Locate the inflection points.

   Ans: Concavity changes at $x = a$ and $x = \dfrac{3a}{2}$; inflection points: $(a, 0)$, $\left(\dfrac{3a}{2}, \dfrac{3a\sqrt[3]{4a}}{4}\right)$

   We check using desmos:



EXAMPLE 8: Below is the graph of the **derivative** of a function. Assume as $x \to \pm\infty$, $f'(x) \to -\infty$.



The graph of $y = f'(x)$

1. Use the graph of $y = f'(x)$ to determine the open intervals where f is increasing and decreasing.

   Find the x-coordinates of the local extrema.

   Ans: local min at $x = 0$; local max at $x = 4$.

2. Use the graph of $y = f'(x)$ make a Sign Diagram for $y = f''(x)$.

   $(+)\ 0\ (-)\quad f''(x)$
   $-\infty \longleftarrow 2 \longrightarrow \infty\quad x$

   $\smile\quad\frown\quad f(x)$
   $-\infty \longleftarrow 2 \longrightarrow \infty\quad x$

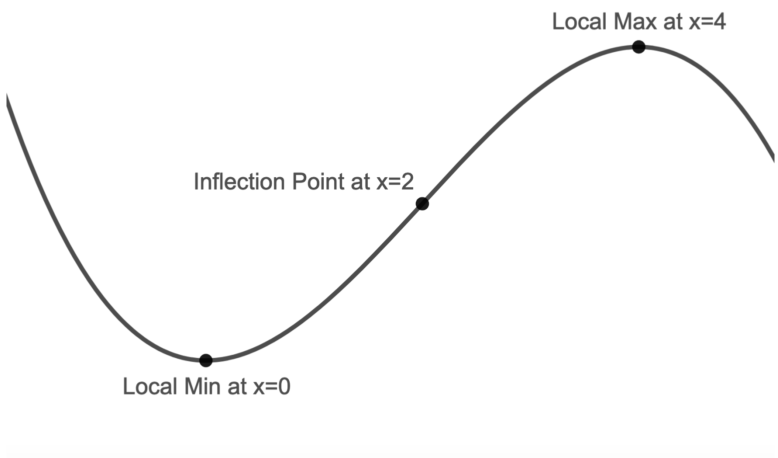
3. List the open intervals over which the graph of f is concave up and concave down.

   Ans: We have f is concave up on $(-\infty, 2)$ and concave down on $(2, \infty)$.

   Find the x-coordinates of the inflection points.

   Ans: Since f changes concavity at $x = 2$, there is an inflection point there.

4. Sketch a plausible graph of $y = f(x)$.



A possible graph of $y = f(x)$

HOMEWORK: Section 4.3: 1 - 93 every other odd